A class of 3-dimensional almost cosymplectic manifolds

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Abstract: The main interest of the present paper is to classify the almost cosymplectic 3-manifolds that satisfy \(\|\text{grad}\lambda\| = \text{const.}(\neq 0)\) and \(\nabla\xi h = 2ah\phi\).

Key words: Almost cosymplectic manifold, cosymplectic manifold

1. Preliminaries

Let \(M\) be an almost contact metric manifold and let \((\phi, \xi, \eta, g)\) be its almost contact metric structure. Thus \(M\) is a \((2n + 1)\)-dimensional differentiable manifold and \(\phi\) is a \((1, 1)\) tensor field, \(\xi\) is a vector field, and \(\eta\) is a 1-form on \(M\), such that

\[\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi)\]
\[\phi(\xi) = 0, \quad \eta \circ \phi = 0,\]
\[g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\]

for any vector fields \(X, Y\) on \(M\).

The fundamental 2-form \(\Phi\) of an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) is defined by

\[\Phi(X, Y) = g(X, \phi Y),\]

for any vector fields \(X, Y\) on \(M\), and this form satisfies \(\eta \wedge \Phi^n \neq 0\). \(M\) is said to be almost cosymplectic if the forms \(\eta\) and \(\Phi\) are closed, that is, \(d\eta = 0\) and \(d\Phi = 0\).

The theory of an almost cosymplectic manifold was introduced by Goldberg and Yano in [9]. The products of almost Kaehler manifolds and the real \(\mathbb{R}\) line or the circle \(S^1\) are the simplest examples of almost cosymplectic manifolds. Topological and geometrical properties of almost cosymplectic manifolds have been studied by many mathematicians (see [4], [11], [5], [9], [15], and [18]).

For \(M\), define \((1, 1)\)-tensor fields \(\tilde{A}\) and \(h\) by ([7],[8],[15],[16])

\[\tilde{A} X = -\nabla X \xi,\]

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\[ h = \frac{1}{2} \mathcal{L}_\xi \phi, \]  

where \( \mathcal{L} \) indicates the Lie differentiation operator and \( \nabla \) is the Levi-Civita connection determined by \( g \). The tensors \( \tilde{A} \) and \( h \) are related by

\[ h = \tilde{A} \phi, \quad \tilde{A} = \phi h. \]  

The main algebraic properties of \( \tilde{A} \) and \( h \) are the following:

\[ g(\tilde{A} X, Y) = g(\tilde{A} Y, X), \quad \tilde{A} \phi + \phi \tilde{A} = 0, \quad \tilde{A} \xi = 0, \quad \eta \circ \tilde{A} = 0, \]

\[ g(h X, Y) = g(h Y, X), \quad h \phi + \phi h = 0, \quad \tilde{A} h = 0, \quad h \xi = 0, \quad \eta \circ h = 0. \]

The curvature tensor \( R \) of \( M \) is given by

\[ R(X, Y, Z) = [r_X, r_Y]Z - [r_Y, r_X]Z + [r_X, \phi h Y]Z - [r_Y, \phi h X]Z, \]

where \( \kappa, \mu, \nu \) are smooth functions. Contact metric manifolds fulfilling Eq. (8) were investigated in [2], [1], [3], and [12].

This work was inspired by [14] and [13]. We carry on those studies to the 3-dimensional almost cosymplectic manifolds in this paper. The purpose of the present paper is to give a new local classification of 3-dimensional almost cosymplectic manifolds under some conditions. The paper is organized in the following way. Section 2 is devoted to some lemmas related to 3-dimensional almost cosymplectic manifolds for later use. In Section 3, we give our main theorem.

All manifolds considered in this paper are assumed to be connected and of class \( C^\infty \).

2. Three-dimensional almost cosymplectic manifolds

Now we shall give some essential Lemmas and notations.

**Lemma 2.1** [10] Let \( M \) be a smooth manifold \( f : M \to \mathbb{R} \) be a smooth real function. Let \( V_1 \) and \( V_2 \) be open sets of \( M \) defined by

\[ V_1 = \{ m \in M \mid f(m) \neq 0 \text{ in a neighborhood of } m \}, \]

\[ V_2 = \{ m \in M \mid f(m) = 0 \text{ in a neighborhood of } m \}. \]

Then \( V_1 \cup V_2 \) is open and dense in \( M \).

Let \( (M, \phi, \xi, \eta, g) \) be an almost cosymplectic 3-manifold. Let

\[ U = \{ p \in M \mid h(p) \neq 0 \text{ in a neighborhood of } p \} \subset M, \]

\[ U_0 = \{ p \in M \mid h(p) = 0 \text{ in a neighborhood of } p \} \subset M. \]
be open sets of $M$. Using Lemma 2.1, we can say that $U \cup U_0$ is an open and dense subset of $M$, and so any property satisfied in $U_0 \cup U$ is also satisfied in $M$. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of $h$ in a neighborhood of $p$ (this we call a $\phi$-basis).

On $U$, we put $h = \lambda e$, $h\phi e = -\lambda \phi e$, where $\lambda$ is a nonvanishing smooth function assumed to be positive.

**Lemma 2.2** [17] On the open set $U$ we have

\[
\begin{align*}
\nabla_\xi e &= -ae, \\
\nabla_\xi \phi e &= be, \\
\nabla_\xi \phi e &= -ce + \lambda \xi,
\end{align*}
\]

where $a$ is a smooth function,

\[
b = \frac{1}{2\lambda} ((\phi e)(\lambda) + A) \quad \text{with} \quad A = \sigma(e) = Ric(e, \xi),
\]

\[
c = \frac{1}{2\lambda} (e(\lambda) + B) \quad \text{with} \quad B = \sigma(\phi e) = Ric(\phi e, \xi),
\]

and $s$ is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$, and $s\phi e = -\phi e$, and $Ric$ is Ricci tensor field.

By Lemma 2.2, we can prove that

\[
\begin{align*}
[e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e, \\
[e, \xi] &= \nabla_e \xi - \nabla_\xi e = (a - \lambda)\phi e, \\
[\phi e, \xi] &= \nabla_{\phi e} \xi - \nabla_\phi \xi = -\lambda e.
\end{align*}
\]

If we adapt Theorem 7 of [17] to a 3-dimensional almost cosymplectic manifolds, we get the following:

**Lemma 2.3** [17] Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold. If $\sigma \equiv 0$, then the $(\kappa, \mu, \nu)$-structure always exists on every open and dense subset of $M$. This means that the Riemannian curvature tensor $R$ of $M$ satisfies

\[
R(X, Y)\xi = -\lambda^2(\eta(Y)X - \eta(X)Y) + 2a(\eta(Y)hX - \eta(X)hY) + \frac{\xi(\lambda)}{\lambda}(\eta(Y)\phi hX - \eta(X)\phi hY),
\]

for all vector fields $X$ and $Y$ on $M$.

### 3. Main theorem and proof

In this section, we will give our main theorem and prove it.

**Theorem 3.1** (Main theorem) Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold with $\|\text{grad} \, \lambda\| = 1$ and $\nabla_\xi h = 2ah\phi$. Then at any point $p \in M$ there exists a chart $(U, (x, y, z))$ such that $\lambda = f(z) \neq 0$ and...
In the first case \((A = Ric(e, \xi) = 0, B = Ric(\varphi e, \xi) = F(y, z))\), the following are valid:

\[
\xi = \frac{\partial}{\partial x}, \quad \varphi e = \frac{\partial}{\partial y} \quad \text{and} \quad e = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + k_3 \frac{\partial}{\partial z}, \quad k_3 \neq 0.
\]

In the second case \((A = Ric(e, \xi) = F(y, z), B = Ric(\varphi e, \xi) = 0)\), the following are valid:

\[
\xi = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y} \quad \text{and} \quad \varphi e = k'_1 \frac{\partial}{\partial x} + k'_2 \frac{\partial}{\partial y} + k'_3 \frac{\partial}{\partial z}, \quad k'_3 \neq 0,
\]

where

\[
k_1(x, y, z) = r(z) = k'_1(x, y, z),
k_2(x, y, z) = k'_2(x, y, z) = 2xf(z) - \frac{(H(y, z) + y)}{2f(z)} + \beta(z),
k_3(x, y, z) = k'_3(x, y, z) = t(z) + \delta, \quad \frac{\partial H(y, z)}{\partial y} = F(y, z),
\]

and \(r, \beta\) are smooth functions of \(z\) and \(\delta\) is constant. Furthermore, \(f(z) = \int k_3(z) dz\).

**Proof.** By virtue of Lemma 2.2, it can be easily proven that the assumption \(\nabla_\xi h = 2ah\varphi\) is equivalent to \(\xi(\lambda) = 0\). From the definition of a gradient of a differentiable function, we get

\[
\text{grad} \lambda = e(\lambda)e + (\varphi e)(\lambda)\varphi e + \xi(\lambda)\xi
\]

Using Eq. (18) and \(\|\text{grad} \lambda\| = 1\) we have

\[
(e(\lambda))^2 + ((\varphi e)(\lambda))^2 = 1.
\]

Differentiating (19) with respect to \(\xi\) and using Eqs. (16) and (17) and \(\xi(\lambda) = 0\), we obtain

\[
\xi(e(\lambda))e(\lambda) + \xi((\varphi e)(\lambda))(\varphi e)(\lambda) = 0,
\]

\[
(\xi, e)(\lambda)e(\lambda) + (\xi, \varphi e)(\lambda)(\varphi e)\lambda = 0,
\]

\[
\lambda e(\lambda)(\varphi e)(\lambda) = 0
\]

and since \(\lambda \neq 0\),

\[
e(\lambda)(\varphi e)(\lambda) = 0.
\]

To study this system, we consider the open subsets of \(U:\)

\[
U' = \{p \in U \mid e(\lambda)(p) \neq 0, \text{ in a neighborhood of } p\},
\]

\[
U'' = \{p \in U \mid (\varphi e)(\lambda)(p) \neq 0, \text{ in a neighborhood of } p\}.
\]

From Lemma 2.1 we have that \(U' \cup U''\) is open and dense in the closure of \(U\). We distinguish 2 cases.
Case 1: We suppose that \( p \in U' \). By virtue of Eqs. (19) and (20), we have \( (\phi e)(\lambda) = 0 \), and \( e(\lambda) = \mp 1 \). Changing to the basis \((\xi, -e, -\phi e)\) if necessary, we can assume that \( e(\lambda) = 1 \). The Eqs. (15), (16), (17), and (13), Eq. (14) reduces to

\[
\begin{align*}
[e, \phi e] &= -be + c\phi e \\
[e, \xi] &= -2\lambda\phi e \\
[\phi e, \xi] &= 0, \quad \lambda = -a \\
b &= \frac{A}{2\lambda}, \quad c = \frac{B + 1}{2\lambda}, \quad a = -\lambda,
\end{align*}
\]

respectively.

Since \([\phi e, \xi] = 0\), the distribution that is spanned by \( \phi e \) and \( \xi \) is integrable, and so for any \( p \in U' \) there exists a chart \( \{V, (x, y, z)\} \) at \( p \), such that

\[
\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + k_3 \frac{\partial}{\partial z}
\]

where \( k_1, k_2, k_3 \) are smooth functions on \( V \). Since \( e, \phi e \) are linearly independent we have \( k_3 \neq 0 \) at any point of \( V \).

Using Eqs. (21), (22) and (25), we get the following partial differential equations:

\[
\begin{align*}
\frac{\partial k_1}{\partial y} &= \frac{A}{2\lambda} k_1, \quad \frac{\partial k_2}{\partial y} = \frac{1}{2\lambda} [Ak_2 - B - 1], \quad \frac{\partial k_3}{\partial y} = \frac{A}{2\lambda} k_3, \\
\frac{\partial k_1}{\partial x} &= 0, \quad \frac{\partial k_2}{\partial x} = 2\lambda, \quad \frac{\partial k_3}{\partial x} = 0.
\end{align*}
\]

Moreover, we know that

\[
\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0.
\]

Differentiating the equation \( \frac{\partial k_3}{\partial x} = 0 \) with respect to \( \frac{\partial}{\partial y} \), and using \( \frac{\partial k_3}{\partial y} = \frac{A}{2\lambda} k_3 \), we find

\[
0 = \frac{\partial^2 k_3}{\partial y \partial x} = \frac{\partial^2 k_3}{\partial x \partial y} = \frac{1}{2\lambda} \frac{\partial A}{\partial x} k_3 + \frac{1}{2\lambda} A \frac{\partial k_3}{\partial x} = \frac{1}{2\lambda} \frac{\partial A}{\partial x} k_3.
\]

So,

\[
\frac{\partial A}{\partial x} = 0.
\]

Differentiating \( \frac{\partial k_2}{\partial x} = 2\lambda \) with respect to \( \frac{\partial}{\partial y} \), and using \( \frac{\partial k_2}{\partial y} = \frac{1}{2\lambda} [Ak_2 - B - 1] \) and Eq. (29), we prove that

\[
\frac{\partial^2 k_2}{\partial y \partial x} = 0 = \frac{\partial^2 k_2}{\partial x \partial y} = \frac{1}{2\lambda} \left[ \frac{\partial A}{\partial x} k_2 + A \frac{\partial k_2}{\partial x} - \frac{\partial B}{\partial x} \right].
\]

So,

\[
\frac{\partial B}{\partial x} = 2\lambda A.
\]
From Eq. (28) we have the following solution:

\[ \lambda(z) = f(z) + d, \quad (31) \]

where \( d \) is constant. For the sake of shortness, we will use \( \tilde{f}(z) \) instead of \( f(z) + d \). Using \( e(\lambda) = k_1 \frac{\partial \lambda}{\partial z} + k_2 \frac{\partial \lambda}{\partial y} + k_3 \frac{\partial \lambda}{\partial z} = 1 \) and Eq. (28), we get

\[ \frac{\partial \lambda}{\partial z} = \frac{1}{k_3}, \quad k_3 \neq 0. \quad (32) \]

If we differentiate Eq. (32) with respect to \( \frac{\partial}{\partial y} \) because of the equation \( \frac{\partial \lambda}{\partial y} = 0 \), we obtain

\[ 0 = \frac{\partial^2 \lambda}{\partial z \partial y} = \frac{\partial^2 \lambda}{\partial y \partial z} = -\frac{1}{k_3^2} \frac{\partial k_3}{\partial y}. \quad (33) \]

Since \( k_3 \neq 0 \), Eq. (33) reduces and then we obtain

\[ \frac{\partial k_3}{\partial y} = 0. \quad (34) \]

Combining Eqs. (26) and (34), we deduced that

\[ A = 0. \quad (35) \]

Using Eqs. (30) and (35), we have

\[ \frac{\partial B}{\partial x} = 0. \quad (36) \]

It follows from Eq. (36) that

\[ B = F(y, z). \quad (37) \]

By virtue of Eqs. (35), (26), and (27), we easily see that

\[ k_1 = r(z), \quad (38) \]

where \( r(z) \) is an integration function.

Combining Eqs. (27) and (34), we get

\[ k_3 = t(z) + \delta, \quad (39) \]

where \( \delta \) is constant.

If we use Eqs. (27), (31), (35), and (37) in Eq. (26),

\[ \frac{\partial k_2}{\partial x} = 2 \tilde{f}(z), \quad \frac{\partial k_2}{\partial y} = \frac{-(B + 1)}{2\lambda} = \frac{-(F(y, z) + 1)}{2f(z)}. \quad (40) \]

It follows from this last partial differential equation that

\[ k_2 = 2x \tilde{f}(z) - \frac{(H(y, z) + y)}{2f(z)} + \beta(z), \quad (41) \]
where
\[
\frac{\partial H(y, z)}{\partial y} = F(y, z) \tag{42}
\]

Because of Eq. (32), there is a relation between \(\lambda(z) = \tilde{f}(z)\) and \(k_3(z)\) such that \(\tilde{f}(z) = \int \frac{1}{k_3(z)} dz\). We will calculate the tensor fields \(\eta, \phi, g\) with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\). For the components \(g_{ij}\) of the Riemannian metric \(g\), using Eq. (25) we have
\[
g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -\frac{k_1}{k_3},
\]
\[
g_{23} = g_{32} = -\frac{k_2}{k_3}, \quad g_{33} = \frac{1 + k_1^2 + k_2^2}{k_3^2}.
\]
The components of the tensor field \(\phi\) are immediate consequences of
\[
\phi(\xi) = \phi\left(\frac{\partial}{\partial x}\right) = 0, \quad \phi\left(\frac{\partial}{\partial y}\right) = -k_1 \frac{\partial}{\partial x} - k_2 \frac{\partial}{\partial y} - k_3 \frac{\partial}{\partial z},
\]
\[
\phi\left(\frac{\partial}{\partial z}\right) = \frac{k_1 k_2}{k_3} \frac{\partial}{\partial x} + \frac{1 + k_2^2}{k_3} \frac{\partial}{\partial y} + k_2 \frac{\partial}{\partial y}.
\]
The expression of the 1-form \(\eta\) immediately follows from \(\eta(\xi) = 1, \eta(\xi) = \eta(\xi) = 0\).
\[
\eta = dx - \frac{k_1}{k_3} dz.
\]
Now we calculate the components of tensor field \(h\) with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\).
\[
h(\xi) = h\left(\frac{\partial}{\partial x}\right) = 0, \quad h\left(\frac{\partial}{\partial y}\right) = -\lambda \frac{\partial}{\partial y},
\]
\[
h\left(\frac{\partial}{\partial z}\right) = \lambda \frac{k_1}{k_3} \frac{\partial}{\partial x} + 2\lambda \frac{k_2}{k_3} \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial z}.
\]

**Case 2:** Now we suppose that \(p \in U''\). As in Case 1, we can assume that \((\phi e)(\lambda) = 1\). The Eqs. (15), (16), (17), and (13), Eq. (14) reduces to
\[
[e, \phi e] = -be + c\phi e, \tag{43}
\]
\[
[e, \xi] = 0, \tag{44}
\]
\[
[\phi e, \xi] = -2\lambda e, \tag{45}
\]
\[
b = \frac{A + 1}{2\lambda}, \quad c = \frac{B}{2\lambda}, \quad a = \lambda, \tag{46}
\]
respectively. Because of Eq. (44), we find that there exists a chart \(\{V', (x, y, z)\}\) at \(p \in U''\) such that
\[
\xi = \frac{\partial}{\partial x}, \quad \phi e = k_1' \frac{\partial}{\partial x} + k_2' \frac{\partial}{\partial y} + k_3' \frac{\partial}{\partial z}, \quad e = \frac{\partial}{\partial y}, \tag{47}
\]

where $k'_1$, $k'_2$, and $k'_3$ ($k'_i \neq 0$), are smooth functions on $V'$.

Using Eqs.(43), (45), and (47), we get the following partial differential equations:

$$\frac{\partial k'_1}{\partial y} = \frac{B}{2\lambda} k'_1, \quad \frac{\partial k'_2}{\partial y} = \frac{1}{2\lambda} [B k'_2 - A - 1], \quad \frac{\partial k'_3}{\partial y} = \frac{B}{2\lambda} k'_3,$$

\begin{align*}
\frac{\partial k'_1}{\partial x} &= 0, \quad \frac{\partial k'_2}{\partial x} = 2\lambda, \quad \frac{\partial k'_3}{\partial x} = 0. 
\end{align*}

Moreover, we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0.$$  

As in Case 1, if we solve the partial differential equations Eq. (48) and Eq. (49), then we find

$$B = 0, \quad A = F'(y,z)$$

$$\lambda(z) = f'(z) + d' = \tilde{f}'(z), \quad k'_1 = r'(z), \quad k'_3 = t'(z) + \delta'$$

$$k'_2 = 2x \tilde{f}'(z) - \frac{(H'(y,z) + y)}{2f(z)} + \beta'(z)$$

$$\frac{\partial H'(y,z)}{\partial y} = F'(y,z)$$

where $d'$ and $\delta'$ are constants.

By the help of Eq. (51), the equation $(\phi e)(\lambda) = 1$ implies

$$\lambda(z) = \tilde{f}'(z) = \int \frac{1}{k'_3(z)} dz.$$

As in Case 1, we can directly calculate the tensor fields $g, \phi, \eta$, and $h$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$g = \begin{pmatrix} 1 & 0 & -\frac{k'_1}{k'_3} \\ 0 & 1 & -\frac{k'_2}{k'_3} \\ -\frac{k'_1}{k'_3} & \frac{k'_2}{k'_3} & 1 \pm \frac{k'_1^2 + k'_2^2}{k'_3^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & k'_1 & -\frac{k'_1 k'_2}{k'_3} \\ 0 & k'_2 & -\frac{1 + k'_2^2}{k'_3} \\ 0 & k'_3 & -k'_2 \end{pmatrix},$$

$$\eta = dx - \frac{k'_1}{k'_3} dz \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & -\lambda \frac{k'_1}{k'_3} \\ 0 & \lambda & -2\lambda \frac{k'_2}{k'_3} \\ 0 & 0 & -\lambda \end{pmatrix}.$$
The 1-form $\eta = dx - zdz$ is closed and the characteristic vector field is $\xi = \frac{\partial}{\partial x}$. Let $g$, $\phi$ be the Riemannian metric and the $(1,1)$-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_2 \\ -a_1 & a_2 & 1 + a_1^2 + (a_2)^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & a_1 & a_1a_2 \\ 0 & -a_2 & -1 + (1 + a_2^2)^2 \\ 0 & 1 & a_2 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 0 & -\lambda a_1 \\ 0 & \lambda & 2\lambda a_2 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda = z,$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a_1 = z$ and $a_2 = 1 - 2xz$.

$$\eta = dx - zdz, \quad d\eta = 0,$$

$$\Phi = -dy \land dz, \quad d\Phi = 0.$$

By a straightforward calculation, we obtain

$$\nabla \xi h = 2zh\phi, \quad F(y, z) = -1, \quad \|\text{grad } \lambda\| = 1.$$

**Remark 3.3** Let $M(\phi, \xi, \eta, g)$ be an almost cosymplectic manifold. A $D_\alpha$-homothetic transformation [19] is the transformation

$$\bar{\eta} = \alpha \eta, \quad \bar{\xi} = \frac{1}{\alpha} \xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$$

of the structure tensors, where $\alpha$ is a positive constant. It is well known [19] that $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost cosymplectic manifold. When 2 contact structures $(\phi, \xi, \eta, g)$ and $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ are related by Eq. (54), we will say that they are $D_\alpha$-homothetic. We can easily show that $h = \frac{1}{\alpha}h$ so $\bar{\lambda} = \frac{1}{\alpha}\lambda$.

(a) As a result, an almost cosymplectic manifold with $\|\text{grad } \lambda\|_g = d \neq 0$ (const.) is $D_\alpha$-deformed in another almost cosymplectic manifold with $\|\text{grad } \lambda\| = d\alpha^{-\frac{3}{2}}$ and choosing $\alpha = d^2$, it is enough to study those almost cosymplectic manifolds with $\|\text{grad } \lambda\| = 1$.

(b) If $d = 0$, then $\lambda$ is constant. As a result, if $\lambda = 0$, then $M$ is a cosymplectic manifold.

**Remark 3.4** There are no compact 3-dimensional almost cosymplectic manifolds with $\|\text{grad } \lambda\| = \text{const} \neq 0$. In fact, if such a manifold is compact, then the smooth function $\lambda$ will attain a maximum value at some point $p$ of $M$. Then $\text{grad } \lambda$ vanishes at $p$, contrary to the requirement that $\text{grad } \lambda$ is a nonzero constant.

**Remark 3.5** Using Theorem 3.1, we can produce infinitely many possible examples about 3-dimensional almost cosymplectic manifolds. If we add the condition $F(y, z) = 0$ to Theorem 3.1, we have $A = 0$ and $B = 0$. Thus, by Lemma 2.3, we can state that a 3-dimensional almost cosymplectic manifold under the same conditions of Theorem 3.1 is a 3-dimensional almost cosymplectic ($\kappa, \mu$) manifold.

Now we will give an example satisfying Remark 3.5.
**Example 3.6** We consider the 3-dimensional manifold

\[ M = \{(x, y, z) \in R^3 \mid z > 0\} \]

and the vector fields

\[ \xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = z^2 \frac{\partial}{\partial x} + (2xz - z + y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \]

The 1-form \( \eta = dx - z^2 dz \) is closed and the characteristic vector field is \( \xi = \frac{\partial}{\partial x} \). Let \( g, \phi \) be the Riemannian metric and the \((1,1)\)-tensor field given by

\[
\begin{pmatrix}
1 & 0 & -\frac{a_1}{a_3} \\
0 & 1 & -\frac{a_2}{a_3} \\
-\frac{a_1}{a_3} & -\frac{a_2}{a_3} & \frac{1 + a_1^2 + a_2^2}{a_3^2}
\end{pmatrix},
\begin{pmatrix}
0 & -a_1 & \frac{a_1 a_2}{a_3} & \frac{a_3}{a_2} \\
0 & -a_2 & \frac{1 + a_1^2 + a_2^2}{a_3^2} & \frac{a_3}{a_2} \\
0 & -a_3 & \frac{2 a_1}{a_2} & \frac{a_3}{a_2} \\
0 & 0 & \frac{a_3}{a_2} & \lambda
\end{pmatrix},
\]

\[ \eta = dx - \frac{a_1}{a_3} dz, \quad h = \left( \begin{array}{ccc}
0 & 0 & \frac{a_3}{a_2} \\
0 & -\lambda & 2 \frac{a_1}{a_2} \\
0 & 0 & \lambda
\end{array} \right) \]

with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \), where \( a_1 = z^2, \ a_2 = 2xz - \frac{z + y}{z}, \ a_3 = 1, \ \lambda = z \).

\[ \eta = dx - z^2 dz, \quad d\eta = 0, \]

\[ \Phi = dy \wedge dz, \quad d\Phi = 0. \]

By direct computations, we get

\[ \|\text{grad} \lambda\| = 1, \nabla \xi h = -2z h \phi, \quad F(y, z) = 0 \]

and

\[ R(X, Y)\xi = (-z^2)(\eta(Y)X - \eta(X)Y) - 2z(\eta(Y)hX - \eta(X)hY) \]

for any vector field \( X, Y \) on \( M \).

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**References**


